THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH2010D Advanced Calculus 2019-2020

Solution to Problem Set 5

1. Can the function $f(x,y) = \dfrac{\sin x \sin^3 y}{1-\cos(x^2+y^2)}$ be defined at $(0,0)$ in such a way that it becomes continuous there?

   Ans:

   Let $\gamma_1(t) = (t,0)$, for $t \in \mathbb{R}$. Then,

$$\begin{aligned}\lim_{t\to 0} f(\gamma_1(t)) &= \lim_{t\to 0} 0 \\ &= 0\end{aligned}$$

   Also, let $\gamma_2(t) = (t,t)$, for $t \in \mathbb{R}$. Then,

$$\begin{aligned}\lim_{t\to 0} f(\gamma_2(t)) &= \lim_{t\to 0} \frac{\sin^4 t}{1-\cos(2t^2)} \\ &= \lim_{t\to 0} \frac{\sin^4 t}{2\sin^2(t^2)} \\ &= \lim_{t\to 0} \frac{1}{2}\cdot\left(\frac{\sin t}{t}\right)^4 \cdot \left(\frac{t^2}{\sin(t^2)}\right)^2 \\ &= \frac{1}{2}\end{aligned}$$

   Therefore, $\lim_{t\to 0} f(\gamma_1(t)) \neq \lim_{t\to 0} f(\gamma_2(t))$ and $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

   As a result, there is no way for us to redefine the function at $(0,0)$ so that it is continuous at that point.

2. Let D be a path connected subset of $\mathbb{R}^n$ and let $f : D \to \mathbb{R}$ be a continuous function.

   Suppose that $\mathbf{a}, \mathbf{b} \in D$ such that $f(\mathbf{a}) < f(\mathbf{b})$.

   Show that for all $L \in \mathbb{R}$ with $f(\mathbf{a}) < L < f(\mathbf{b})$, there exists $\mathbf{c} \in D$ such that $f(\mathbf{c}) = L$.

   Ans:

   Let $L \in \mathbb{R}$ such that $f(\mathbf{a}) < L < f(\mathbf{b})$. Since D is path connected, there exists a continuous function $\gamma : [0,1] \to D$ such that $\gamma(0) = \mathbf{a}$ and $\gamma(1) = \mathbf{b}$.

   Let $g : [0,1] \to \mathbb{R}$ be a function defined by $g(t) = (f \circ \gamma)(t) = f(\gamma(t))$.

   Note that g is a continuous function and $g(0) = f(\mathbf{a}) < L < f(\mathbf{b}) = g(1)$. By intermediate value theorem, there exists $t_0 \in (0,1)$ such that $g(t_0) = f(\gamma(t_0)) = L$.

   Let $\mathbf{c} = \gamma(t_0)$. Then, $\mathbf{c}$ is a point in D such that $f(\mathbf{c}) = L$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and let $(a,b) \in \mathbb{R}^2$. We define two single variable functions $g(x) = f(x,b)$ and $h(y) = f(a,y)$.

   (a) If $g(x)$ is continuous at $x = a$ and $h(y)$ is continuous at $y = b$, does it follow that f is continuous at (a,b)? Why?

   (b) If $f(x,y)$ is continuous at (a,b), does it follow that $g(x)$ is continuous at $x = a$ and $h(y)$ is continuous at $y = b$? Why?

   Ans:

(a) No, consider the function

$$f(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 0; \\ \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that f is not continuous at $(0,0)$ as $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

However, $g(x) = f(x,0) = 1$ and $h(y) = f(0,y) = 1$ which are continuous functions.

(b) Yes. Since f is continuous at (a,b), given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x,y) - f(a,b)| < \epsilon$ for all $|(x,y) - (a,b)| < \delta$.

Then, for all $|x - a| < \delta$, we have $|(x,b) - (a,b)| = |x - a| < \delta$ and so $|g(x) - g(a)| = |f(x,b) - f(a,b)| < \epsilon$. Therefore, g is continuous at $x = a$.

Similarly, for all $|y-b| < \delta$, we have $|(a,y)-(a,b)| = |y-b| < \delta$ and so $|h(y)-h(b)| = |f(a,y)-f(a,b)| < \epsilon$. Therefore, h is continuous at $y = b$.

4. Let $f(x,y) = \sqrt{2x + 3y - 1}$. Using the limit definition, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(-2,3)$.

Ans:

$$\begin{aligned} \lim_{h\to 0} \frac{f(-2+h,3) - f(-2,3)}{h} &= \lim_{h\to 0} \frac{\sqrt{2h+4} - 2}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2h+4} - 2}{h} \cdot \frac{\sqrt{2h+4} + 2}{\sqrt{2h+4} + 2} \\ &= \lim_{h\to 0} \frac{2}{\sqrt{2h+4} + 2} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, $\dfrac{\partial f}{\partial x}(-2,3) = \dfrac{1}{2}$.

$$\begin{aligned} \lim_{h\to 0} \frac{f(-2,3+h) - f(-2,3)}{h} &= \lim_{h\to 0} \frac{\sqrt{3h+4} - 2}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{3h+4} - 2}{h} \cdot \frac{\sqrt{3h+4} + 2}{\sqrt{3h+4} + 2} \\ &= \lim_{h\to 0} \frac{3}{\sqrt{3h+4} + 2} \\ &= \frac{3}{4} \end{aligned}$$

Therefore, $\dfrac{\partial f}{\partial y}(-2,3) = \dfrac{3}{4}$.

5. Let $f(x,y,z) = xy + yz + zx$. Using the limit definition, find the directional derivative of f at the point $\mathbf{u} = (1,-1,1)$ along the direction $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

Ans:

Note that the unit vector of $\mathbf{v}$ is $\hat{\mathbf{v}} = \dfrac{1}{|\mathbf{v}|}\mathbf{v} = \dfrac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$$\begin{aligned}
& \lim_{h\to 0} \frac{f(\mathbf{x}_0 + h\hat{\mathbf{v}}) - f(\mathbf{x}_0)}{h} \\
= & \lim_{h\to 0} \frac{f(1+\frac{1}{\sqrt{6}}h, -1+\frac{2}{\sqrt{6}}h, 1+\frac{1}{\sqrt{6}}h) - f(1,-1,1)}{h} \\
= & \lim_{h\to 0} \frac{[(1+\frac{1}{\sqrt{6}}h)(-1+\frac{2}{\sqrt{6}}h) + (-1+\frac{2}{\sqrt{6}}h)(1+\frac{1}{\sqrt{6}}h) + (1+\frac{1}{\sqrt{6}}h)(1+\frac{1}{\sqrt{6}}h)] - [(1)(-1)+(-1)(1)+(1)(1)]}{h} \\
= & \lim_{h\to 0} \frac{\frac{5}{6}h^2 + \frac{4}{\sqrt{6}}h}{h} \\
= & \lim_{h\to 0} \frac{4}{\sqrt{6}} + \frac{5}{6}h \\
= & \frac{4}{\sqrt{6}} \\
= & \frac{2\sqrt{6}}{3}
\end{aligned}$$

Therefore, $\nabla_{\mathbf{v}} f(\mathbf{u})$ exists and it equals to 4.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by

$$f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0); \\ \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(0,0)$.

Ans:

We have

$$\lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sin h^3}{h^3} = 1$$

Therefore, $\dfrac{\partial f}{\partial x}(0,0) = 1$. Also

$$\lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sin h^4}{h^3} = \lim_{h\to 0} \frac{\sin h^4}{h^4} \cdot h = \left(\lim_{h\to 0} \frac{\sin h^4}{h^4}\right) \cdot \left(\lim_{h\to 0} h\right) = 1 \cdot 0 = 0$$

Therefore, $\dfrac{\partial f}{\partial y}(0,0) = 0$.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0); \\ \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

(a) Show that $\dfrac{\partial f}{\partial y}(x,0) = x$ for all $x \in \mathbb{R}$ and $\dfrac{\partial f}{\partial x}(0,y) = -y$ for all $y \in \mathbb{R}$.

(b) Show that $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) \neq \dfrac{\partial^2 f}{\partial y \partial x}(0,0)$.

Ans:

(a) If $x \neq 0$, we have $\displaystyle\lim_{h\to 0} \frac{f(x,0+h) - f(x,0)}{h} = \lim_{h\to 0} x\left(\frac{x^2-h^2}{x^2+h^2}\right) = x.$

On the other hand, we have $\displaystyle\lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0.$

Combining the above two cases, $\dfrac{\partial f}{\partial y}(x,0) = x$ for all $x \in \mathbb{R}$.

Also, if $y \neq 0$, we have $\lim_{h\to 0} \dfrac{f(0+h,y)-f(0,y)}{h} = \lim_{h\to 0} y\left(\dfrac{h^2-y^2}{h^2+y^2}\right) = -y$.

On the other hand, we have $\lim_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \dfrac{0-0}{h} = \lim_{h\to 0} 0 = 0$.

Combining the above two cases, $\dfrac{\partial f}{\partial x}(0,y) = -y$ for all $y \in \mathbb{R}$.

(b) We have $\lim_{h\to 0} \dfrac{\frac{\partial f}{\partial x}(0,0+h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \dfrac{-h-0}{h} = -1$, so $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) = -1$.

Also, $\lim_{h\to 0} \dfrac{\frac{\partial f}{\partial y}(0+h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0} \dfrac{h-0}{h} = 1$, so $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = 1$.

Therefore, $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) = -1 \neq 1 = \dfrac{\partial^2 f}{\partial y \partial x}(0,0)$.

8. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ if

(a) $f(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right)$

(b) $f(x,y) = e^{xy}\ln y$

Ans:

(a) $\dfrac{\partial f}{\partial x} = \dfrac{1}{(\frac{y}{x})^2+1} \cdot (-\dfrac{y}{x^2}) = \dfrac{-y}{x^2+y^2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{1}{(\frac{y}{x})^2+1} \cdot (\dfrac{1}{x}) = \dfrac{x}{x^2+y^2}$

(b) $\dfrac{\partial f}{\partial x} = ye^{xy}\ln y$ and $\dfrac{\partial f}{\partial y} = xe^{xy}\ln y + e^{xy}(\dfrac{1}{y}) = e^{xy}\left(x\ln y + \dfrac{1}{y}\right)$

9. Find all first partial derivatives if $f(x,y,z) = \sin^{-1}(x^2+y^2z)$.

Ans:

$\dfrac{\partial f}{\partial x} = \dfrac{2x}{\sqrt{1-(x^2+y^2z)^2}}$, $\dfrac{\partial f}{\partial y} = \dfrac{2yz}{\sqrt{1-(x^2+y^2z)^2}}$ and $\dfrac{\partial f}{\partial z} = \dfrac{y^2}{\sqrt{1-(x^2+y^2z)^2}}$.

10. If $f(x,y) = x\cos y + ye^x$, find all the second-order derivatives, i.e. $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$.

Ans:

$\dfrac{\partial^2 f}{\partial x^2} = ye^x$, $\dfrac{\partial^2 f}{\partial y^2} = -x\cos y$ and $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = -\sin y + e^x$.